Halpern's Iteration Process for Multiple Sets Split Common Fixed Point of Quasi-Nonexpansive Mappings (Study on Nonlinear Analysis and Convex Analysis)

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Abstract

In this paper, we consider iteration processes of Halpern's type to find fixed point of quasi-nonexpansive mapping and common element of solution for the split common fixed point of quasi-nonexpansive mappings. We establish strong convergence theorems of this problems. We apply our results to study the common element of solution of multiple split fixed point problems for quasi-nonexpansive mappings. We also apply our result to study common element of solution for the equilibrium problem and the fixed point of generalized hybrid mapping. Our result gives an partial answer to two open questions which were given by Chidume and Chidume [11], and Kurokawa and Takahashi[12].

Keywords: Fixed point of quasi-nonexpansive mappings, strong quasi-nonexpansive mapping, hybrid mapping, widely more generalized mapping, multiple split fixed point problem, split feasibility problem, multiple sets split feasibility problem

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1 Introduction

Let C, and Q be nonempty closed convex subsets of Hilbert spaces H_1 and H_2 respectively and $A : H_1 \to H_2$ be a bounded linear operator.

The split feasibility problem (**SFP**) is the problem : Find

$\bar{x} \in H_1$ such that $\bar{x} \in C$ and $A\bar{x} \in Q$.

The split feasibility problem (**SFP**) in finite dimensional Hilbert spaces was first introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. The split feasibility problem (**SFP**) has many applications in signal processing, image reconstruction, intensity-modulated radiation therapy, approximation theory, control theory, biomedical engineering, communications, and geophysics. For example, one can see [2, 3, 4, 5].

Let H_1 and H_2 be Hilbert spaces, $U : H_1 \to H_1, T : H_2 \to H_2$ be two operators. Let $Fix(U) = \{x \in H_1 : x = Ux\}$ and $Fix(T) = \{x \in H_2 : x = Tx\}$ be the fixed point sets of U and T respectively.

The split common fixed point problem (**SCFP**) is the problem:

Find $\bar{x} \in H_1$ such that $\bar{x} \in Fix(U)$ and $A\bar{x} \in Fix(T)$.

If H_1 and H_2 are finite dimensional spaces. Censor and Segal[6] propose the following iteration process :

$$x_{n+1} = U(x_n - \lambda A^*(I - T)Ax_n$$

.

Censor and Segal[6]proved that $\{x_n\}$ converges strongly to the solution of (**SCFP**) under suitable assumption.

in 2011,Moudafi [7]established he following weak convergence (**SCFP**) for quasi-nonexpansive mappings.

Theorem 1.1. [7] Let H_1 and H_2 be Hilbert spaces ,$U : H_1 \to H_1, T : H_2 \to H_2$ be two demiclosed quasi-nonexpansive mappings. Suppose that $\Gamma = \{x \in Fix(U), Ax \in Fix(T)\} \neq \emptyset$. Let $x_0 \in H_1$,

$$u_n = x_n - \gamma\beta A^*(I - T)Ax_n,$$

$$x_{n+1} = (1 - \alpha_n)u_n + \alpha_n(x_n - \gamma\beta A^*(I - T)Ax_n),$$

where $\beta \in (0,1), \alpha_n \in (0,1)$, and $\gamma \in (0, \frac{1}{\lambda\beta})$ and $\lambda = \|AA^*\|$. Then $\{x_n\}$ converges weakly to $x^* \in \Gamma$.

In 2014,Kraikaew and Saejung[8] established the following result:

Theorem 1.2. [8] Let H_1 and H_2 be Hilbert spaces and let $U : H_1 \to H_1$ be a strongly quasi-nonexpansive operator, and $T : H_2 \to H_2$ be a quasi-nonexpansive operator such that U and T are demiclosed. Let $A : H_1 \to H_2$be a bounded linear operator. Suppose that $\Gamma = \{x \in Fix(U), Ax \in Fix(T)\} \neq \emptyset$. Let $x_0 \in H_1$ and let $\{x_n\} \subset H_1$ be a sequence defined by

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)U(I - \gamma A^*(I - T)Ax_n),$$

where the parameter and the sequence $\{\alpha_n\}$ satisfies the following conditions:

$(C_1) : \{\alpha_n\} \subset (0,1), \lim\limits_{n\to\infty} \alpha_n = 0$, and

$(C_2) : \sum_{n=0}^{\infty} \alpha_n = \infty$.

$(C_3) : \gamma \in (0, \frac{1}{L})$.

Then $x_n \to P_\Gamma x_0$.

The following strong convergence theorem of Halpern's type[9]was proved by Withmann [10].

Theorem 1.3. [10] Let H_1 be a Hilbert space and let C be a nonempty closed convex subset of H_1 and $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. For any $x_1 = x \in C$, define a sequence $\{x_n\} \in C$ by

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n \text{ for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ satisfies

$(C_1) : \{\alpha_n\} \subset (0,1), \lim\limits_{n\to\infty} \alpha_n = 0$,

$(C_2) : \sum_{n=1}^{\infty} \alpha_n = \infty$,

$(C_3) : \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| = \infty$.

Then $\{x_n\}$ converges strongly to a point $\bar{x} \in Fix(T)$.

Chidume and Chidume [11], give the following question:

Are the conditions $(C_1):\{\alpha_n\}\subset(0,1)$, $\lim\limits_{n\to\infty}\alpha_n=0$, and $(C_2):\sum_{n=1}^{\infty}\alpha_n=\infty$ sufficient for convergence of algorithm of Halpern's type

$$x_{n+1}=\alpha_n u+(1-\alpha_n))Tx_n, n\geq 0$$

for all nonexpansive mapping $T:C\to C$.

Kurokawa and Takahashi[12]proved the strong convergence theorem for nonspreading mapping in Hilbert space:

Theorem 1.4. [12] Let C be a nonempty closed convex subset of a Hilbert space H_1. Let $T:C\to C$ be a nonspreading mapping. Let $u\in C$ and define two sequences $\{x_n\}$ and $\{z_n\}$ as follows: $x_1=x\in C$

$(i)x_{n+1}=\alpha_n u+(1-\alpha_n)z_n$, and

$(ii)z_n=\frac{1}{n}\sum_{k=0}^{n}T^k x_n$

for all $n=1,2,\cdots$, where $\{\alpha_n\}\subset(0,1)$, $\lim\limits_{n\to\infty}\alpha_n=0$, and $\sum\limits_{n=1}^{\infty}\alpha_n=\infty$. If $Fix(T)\neq\emptyset$, then $\{x_n\}$ and $\{z_n\}$ converge strongly to $P_{Fix(T)}u$, where $P_{Fix(T)}$ is the metric projection of H_1 to $Fix(T)$.

Kurokawa and Takahashi[12] gave the following open question:We do not know whether a strong convergence of Halpern's type for nonspreading mapping or not.

Motivated by the above two questions, In this paper,we consider iteration processes of Halpern's type with conditions (C_1) and (C_2) for quasi-nonexpansive mapping , we establish strong convergence theorems to find the fixed point of quasi-nonexpansive mappint with Halpern's iteration process. We also use the Halpern's iteration processes to find the common element of solution for the split common fixed point of quasi-nonexpansive mappings. We establish strong convergence theorems of this problem. We apply our results to study the common element of solution of multiple split fixed point problems for quasi-nonexpansive mappings. We also apply our result to study common element of solution for the equilibrium problem and fixed point of generalized hybrid mapping. Our result gives an partial answer to two open questions which were given by Chidume and Chidume [11], and Kurokawa and Takahashi[12].

2 Preliminaries

Let H_1 be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|| \cdot ||$, respectively. We denote the strongly convergence and the weak convergence of $\{x_n\}_{n\in\mathbb{N}}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let H_1 and H_2 be real Hilbert spaces, let $I_1 : H_1 \to H_1$ be the identity mapping on H_1 and $I_2 : H_2 \to H_2$ be the identity mapping on H_2. Let C be a nonempty, closed, and convex subset of a real Hilbert space H_1, and $T : C \to H_1$ be a mapping. Let $Fix(T) := \{x \in C : Tx = x\}$. Throughout this paper, we use this notations unless specified otherwise. Let C be a nonempty, closed, and convex subset of a real Hilbert space H_1, and $T : C \to H$ be a mapping. Then

(1) T is nonexpansive if $||Tx - Ty|| \leq ||x - y||$ for all $x, y \in C$;

(2) T is quasi-nonexpansive if $Fix(T) \neq \emptyset$ and

$\|Tx - y|| \leq ||x - y||$ for all $x, \in C, y \in Fix(T)$;

(3) T is generalized (α, β) hybrid[13], if $\alpha, \beta \in \mathbb{R}$ and

$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|Ty - x\|^2 \leq (1 - \beta)\|x - y\|^2 + \beta\|Tx - y\|^2$ for all $x, y \in C$;

(4) T is $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ widely more generalized hybrid [14]if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2$
$\leq \varepsilon\|x - Ty\|^2 + \zeta\|y - Ty\|^2 + \eta\|x - Tx - (y - Ty)\|^2$, for all $x, y \in C$;

(5) T is strongly quasi-nonexpansive [15] if $Fix(T) \neq \emptyset$,

$\|Tx - y\| \leq \|x - y\|$ for all $y \in Fix(T)$ and $\|x_n - Tx_n\| \to 0$

whenever $\{x_n\}$ is a bounded sequence in H and $\|x_n - p\| - \|Tx_n - p\| \to 0$ for some $p \in Fix(T)$.

Let C be a nonempty closed convex subset of a real Hilbert space H_1. Let $T : C \to H_1$ be a mapping. T is said to be demiclosed if for each sequence $\{x_n\}$ and x in C with $x_n \rightharpoonup x$ and $(I-T)x_n \to 0$ implies that $(I-T)x = 0$.

We know that the Ky Fan minimax inequality problem is to find $z \in C$ such that

$$(\mathbf{EP}) \quad g(z,y) \geq 0 \text{ for each } y \in C,$$

where $g : C \times C \to \mathbb{R}$ is a bifunction. This problem includes fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, minimax inequalities, and saddle point problems as special cases. (For examples, one can see [16] and related literature.) The solution set of Ky Fan minimax inequality problem (**EP**) is denoted by (**EP(C,g)**).

To solve the Ky Fan minimax inequality problem, we assume that the bifunction $g : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $g(x,x) = 0$ for each $x \in C$;

(A2) g is monotone, i.e., $g(x,y) + g(y,x) \leq 0$ for any $x, y \in C$;

(A3) for each $x, y, z \in C$, $\limsup_{t \downarrow 0} g(tz + (1-t)x, y) \leq g(x,)$;

(A4) for each $x \in C$, the scalar function $y \to g(x,y)$ is convex and lower semicontinuous.

3 Main Results

Theorem 3.1. Let C be a closed convex subset of a Hilbert space H_1, let $\omega \in (0,1)$, and let $T : C \to C$ be a ω-strongly quasi-nonexpansive operator such that T is demiclosed. Let $x_0 \in C$ and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence defined by

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Tx_n,$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then

$$\lim_{n\to\infty} x_n = P_{Fix(T)}x_0.$$

.

Theorem 3.2. Let C be a closed convex subset of a Hilbert space H_1 and let $T : C \to C$ be a quasi-nonexpansive operator such that T is demiclosed. Let $\omega \in (0,1)$, $x_0 \in C$ and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence defined by

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)((1-\omega)I_1 + \omega T)x_n,$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$ is a sequence in $(0,1)$ such that $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then

$$\lim_{n\to\infty} x_n = P_{Fix(T)}x_0.$$

Theorem 3.3. Let $U_i : H_1 \to H_1, i \in \{1, 2, \cdots, m\} = I$ and $S_j : H_2 \to H_2, j \in \{1, 2, \cdots, l\} = J$ be demiclosed quasi-nonexpansive mappings, and Let $A : H_1 \to H_2$ be a bounded linear operator with $\|A\| > 0$. Let $\{\lambda_i : i \in I\}$, and $\{\eta_j : j \in J\}$ be strict positive numbers such that $\{\lambda_i\}_{i\in I} \in \triangle_m$ and $\{\eta_j\}_{j\in J} \in \triangle_l$. Let

$U = \sum_{i=1}^{m} \lambda_i U_{i\omega}$, and $V = I_1 - \frac{1}{\|A\|^2} A^*(I_2 - \sum_{j=1}^{\ell} \eta_j S_{j\omega})A$

$U_{i\omega} = (1-\omega)I_1 + \omega U_i$ and $S_{j\omega} = (1-\omega)I_2 + \omega S_j$.

Suppose that $\Gamma = \{x \in \bigcap_{i=1}^{m} Fix(U_i), Ax \in \bigcap_{j=1}^{\ell} Fix(S_j)\} \neq \emptyset$. Let $x_0 \in H_1$ and let $\{x_n\}_{n\in\mathbb{N}} \subset H_1$ be a sequence defined by

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)UVx_n,$$

where the parameter and the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ satisfies the following conditions:

(i)$\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1)$, $\lim\limits_{n\to\infty} \alpha_n = 0$, and

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then $x_n \to P_\Gamma x_0$.

Theorem 3.4. Let $U_i : H_1 \to H_1, i \in \{1, 2, \cdots, m\} = I$ and $S_j : H_2 \to H_2, j \in \{1, 2, \cdots, \ell\} = J$ be quasi-nonexpansive mappings.

Let $A : H_1 \to H_2$ be a bounded linear operator with $\|A\| > 0$. Suppose that $\Gamma = \{x \in \bigcap_{i=1}^{m} Fix(U_i), Ax \in \bigcap_{j=1}^{l} Fix(S_j)\} \neq \emptyset$. Let $\omega \in (0,1)$,

$U_{i\omega} = (1-\omega)I_1 + \omega U_i$ and $S_{j\omega} = (1-\omega)I_2 + \omega S_j$. $U = U_{1\omega}U_{2\omega}\cdots U_{m\omega}, S = S_{1\omega}S_{2\omega}\cdots S_{l\omega}$, and let $V = I_1 - \frac{1}{\|A\|^2}A^*(I_2 - S_{1\omega}S_{2\omega}\cdots S_{l\omega})A$.

Let $x_0 \in H_1$ and let $\{x_n\}_{n\in\mathbb{N}} \subset H_1$ be a sequence defined by

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)UVx_n,$$

where the parameter and the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ satisfies the following conditions:

(i)$\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1), \subset (0,1), \lim_{n\to\infty} \alpha_n = 0$, and

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then $x_n \to P_\Gamma x_0$.

Theorem 3.5. Let $U_i : H_1 \to H_1, i \in \{1,2,\cdots,m\} = I$ be demiclosed quasi-nonexpansive mappings.

Suppose that $\Gamma = \{x \in \bigcap_{i=1}^{m} Fix(U_i)\} \neq \emptyset$. Let $\omega \in (0,1)$,

$U_{i\omega} = (1-\omega)I_1 + \omega U_i$, $U = U_{1\omega}U_{2\omega}\cdots U_{m\omega}$.

Let $x_0 \in H_1$ and let $\{x_n\}_{n\in\mathbb{N}} \subset H_1$ be a sequence defined by

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)Ux_n,$$

where the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ satisfies the following conditions:

(i)$\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1), \subset (0,1), \lim_{n\to\infty} \alpha_n = 0$, and

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then $x_n \to P_\Gamma x_0$.

Theorem 3.6. Let $U_i : H_1 \to H_1, i \in \{1,2,\cdots,m\} = I$ and $S_j : H_2 \to H_2, j \in \{1,2,\cdots,\ell\} = J$ be demiclosed quasi-nonexpansive mappings , Let $A_j : H_1 \to H_2, j = 1,2,\cdots,\ell$ be bounded linear operators with $\|A_j\| > 0$, let $\Gamma = \{x \in H_1 : x \in \bigcap_{i=1}^{m} Fix(S_i), A_j x \in Fix(S_j)$ for all $j = 1,2,\cdots,\ell\} \neq \emptyset$ .Let $\{\lambda_i : i \in I\}$, and $\{\eta_j : j \in J\}$ be strict positive numbers such that $\{\lambda_i\}_{i\in I} \in \triangle_m$ and $\{\eta_j\}_{j\in J} \in \triangle_l$. Let

$U = \sum_{i=1}^{m} \lambda_i U_{i\omega}$, and $V = \sum_{j=1}^{\ell} \eta_j (I_1 - \frac{1}{\|A_j\|^2} A_j^*(I_2 - S_{j\omega})A_j)$,

where $U_{i\omega} = (1-\omega)I_1 + \omega U$

and

$S_{j\omega} = (1-\omega)I_2 + \omega S_j$.

Suppose that $\Gamma = \{x \in \bigcap_{i=1}^{m} Fix(U_i), A_j x \in Fix(S_j) \text{ for all } j = 1, 2, \cdots, \ell\} \neq \emptyset$.

Let $x_0 \in H_1$ and let $\{x_n\}_{n\in\mathbb{N}} \subset H_1$ be a sequence defined by

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)UVx_n,$$

where the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ satisfies the following conditions:

(i)$\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1)$, $\lim\limits_{n\to\infty} \alpha_n = 0$, and

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then $x_n \to P_\Gamma x_0$.

4 Applications

Theorem 4.1. Let C be a nonempty closed convex subset of H_1. Let $G : C \times C$ be a function satisfying $A_1 - A_4$. Let $U : H_1 \to H_1$ be

$(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ widely more generalized hybrid mapping with $Fix(U) \neq \emptyset$ which satisfies the condition (1) or (2):

(1)$\alpha + \beta + \gamma + \delta \geq 0, \alpha + \beta > 0$ and $\zeta + \eta \geq 0$.

(2)$\alpha + \beta + \gamma + \delta \geq 0, \alpha + \gamma > 0,$ and $\varepsilon + \eta \geq 0$.

Let $\omega \in (0,1), U_\omega = (1-\omega)I_1 + \omega U$ and let

$$T_r^G x = \left\{ z \in C : G(z,y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \ \forall \ y \in C \right\}$$

for all $x \in H$.

Suppose that $\Gamma = Fix(U) \bigcap EP(C,G) \neq \emptyset$.

Let $x_0 \in C$ and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence defined by

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n) U_\omega T_r^G x_n,$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$ is a sequence in $(0,1)$ such that $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then

$$\lim_{n\to\infty} x_n = P_\Gamma x_0.$$

Theorem 4.2. Let $U : H_1 \to H_1$ be

$(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ widely more generalized hybrid mapping with $Fix(T) \neq \emptyset$ which satisfies the condition (1) or (2):

(1)$\alpha + \beta + \gamma + \delta \geq 0, \alpha + \beta > 0$ and $\zeta + \eta \geq 0$.

(2)$\alpha + \beta + \gamma + \delta \geq 0, \alpha + \gamma > 0,$ and $\varepsilon + \eta \geq 0$.

Let $S : C \to H_1$ be a $(\alpha_1, \beta_1, \gamma_1, \delta_1, \varepsilon_1, \zeta_1, \eta_1)$ widely more generalized hybrid mapping with $Fix(T) \neq \emptyset$ which satisfies the condition (3) or (4):

(3)$\alpha_1 + \beta_1 + \gamma_1 + \delta_1 \geq 0, \alpha_1 + \beta_1 > 0$ and $\zeta_1 + \eta_1 \geq 0$.

(4)$\alpha_1 + \beta_1 + \gamma_1 + \delta_1 \geq 0, \alpha_1 + \gamma_1 > 0,$ and $\varepsilon_1 + \eta_1 \geq 0$.

Let $A : H_1 \to H_2$ be a bounded linear operator with $\|A\| > 0$. Suppose that $\Gamma = \{x \in Fix(U), Ax \in Fix(S)\} \neq \emptyset$. Let $\omega \in (0,1)$, and let $V = I_1 - \frac{1}{\|A\|^2} A^*(I_2 - S_\omega)A$. Let $U_\omega = (1-\omega)I_1 + \omega U$ and $S_\omega = (1-\omega)I_2 + \omega S$. Let $x_0 \in H_1$ and let $\{x_n\}_{n\in\mathbb{N}} \subset H_1$ be a sequence defined by

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n) U_\omega V,$$

where the parameter and the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ satisfies the following conditions:

(i)$\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1)$, $\lim\limits_{n\to\infty} \alpha_n = 0$, and

(ii)$\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then $x_n \to P_\Gamma x_0$.

Theorem 4.3. [14] Let $U : H_1 \to H_1$ be

$(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ widely more generalized hybrid mapping with $Fix(U) \neq \emptyset$ which satisfies the condition (1) or (2):

(1)$\alpha+\beta+\gamma+\delta \geq 0, \alpha+\beta>0$ and $\zeta+\eta \geq 0$.

(2)$\alpha+\beta+\gamma+\delta \geq 0, \alpha+\gamma>0,$ and $\varepsilon+\eta \geq 0$.

Suppose that $\Gamma = Fix(U) \neq \emptyset$. Let $U_\omega = (1-\omega)I_1 + \omega U$ for $\omega \in (0,1)$. Let $x_0 \in H_1$ and let $\{x_n\}_{n\in\mathbb{N}} \subset H_1$ be a sequence defined by

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)U_\omega x_n,$$

where the parameter and the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ satisfies the following conditions:

(i)$\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1)$, $\lim\limits_{n\to\infty} \alpha_n = 0$, and

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then $x_n \to P_\Gamma x_0$.

Theorem 4.4. [17] Let C be a nonempty closed convex subset of H_1. Let $T : C \to C$be a (α, β) generalized hybrid mapping with $\alpha < \beta$. Let $\omega \in (0,1), T_\omega = (1-\omega)I_1 + \omega T$.

Suppose that $Fix(T) \neq \emptyset$.

Let $x_0 \in C$ and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence defined by

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)T_\omega x_n,$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$ is a sequence in $(0,1)$ such that $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then

$$\lim_{n\to\infty} x_n = P_{Fix(T)} x_0.$$

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5 Numerical Example

Example 5.1. Let $H_1 = \mathbb{R}, C = [-5, \infty)$. Let $T : C \to C$ be defined by $T(x) = \frac{x-5}{2}, x \in C$. It is easy to see $Fix(T) = \{-5\}$.

$|T(x) - y| = |\frac{x+5}{2}| = \frac{x+5}{2} \leq (x+5) \leq |x+5|$, for all $y \in Fix(T) = \{5\}$.

Therefore T is a quasi-nonexpansive mapping.

Let $\alpha_n = \frac{1}{2n}, \omega = 0.1, x_0 = 1$.

Then

$x_{n+1} = \alpha_n x_0 + (1-\alpha_n(\omega x_n + (1-\omega)Tx_n = \frac{1}{2n} + (1-\frac{1}{2n})\frac{1.1x_n - 4.5}{2}$.

We see

$x_1 = -0.35, x_2 = -1,70685, x_3 = -2.49651, x_4 = -3.0423758, x_5 = -3.430976, x_{10} = -4.2718038, x_{20} = -4.6311819, x_{30} = -4.7726976, x_{40} = -4.8305837, x_{50} = -4.8650305, x_{60} = -4.8877178, x_{70} = -4.9038248, x_{80} = -4.9160061, x_{90} = -4.9254063, x_{100} = -4.9329136, x_{110} = -4.939049, x_{120} = -4.9441544, x_{130} = -4,9484713, x_{140} = -4.9521689, x_{150} = -4.955371, x_{160} = -4.9581712, x_{170} = -4.9606409, x_{180} = -4.9628352, x_{190} = -4.96479977, x_{200} = -4.9665632, x_{210} = -4,9681608, x220 = -4.9696117, x_{223} = -4.9700216.$

From these results, we see $\lim_{n\to\infty} x_n = -5 \in P_{Fix(T)}x_0 = \{-5\}$

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